Mechanics: Vectors and scalars

Objectives

- Identify physical quantities and categorise them as vectors or scalars.
- Define a vector and a scalar.
- Represent vectors graphically with a scale and an arrow.
- Understand equality of vectors, negative vectors and equilibrants.
- Define a resultant vector.
- Add vectors in one dimension and in two dimensions graphically using the head-to-tail method.
- Add vectors in one dimension and in two dimensions (for right angled triangles) algebraically.

Defining vectors and scalars

In our daily life we encounter many physical quantities such as time, mass, weight, force and electric charge. All physical quantities may be divided into two broad groups, scalars and vectors.

A scalar [D] is a physical quantity that has only a magnitude (size).

A vector [D] is a physical quantity that has magnitude (size) and direction.

For example, a tub of margarine is labelled with a mass of 500 g. The mass of the tub of margarine is a scalar quantity. It is only described by its magnitude, which is 500 g. Speed is also a scalar quantity. For example, you walked to the shops at a speed of 4 $km.h^{-1}$ (magnitude).

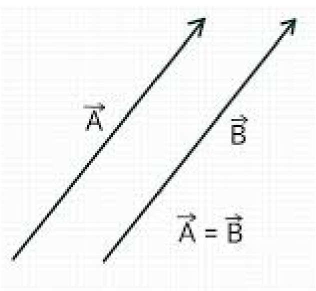
However, velocity, which is similar to speed, is a vector. Therefore your velocity was 4 $km.h^{-1}$ (magnitude) towards the northeast (direction). The answer is incomplete without including the direction.

Most physical quantities are scalars. The following physical quantities are the most important vectors that we need to be aware of:

- Force, $\vec{F}$, measured in N.
- Weight (a type of force that always acts down), measured in N.
- Displacement, $\vec{x}$, measured in m.
- Velocity, $\vec{v}$, measured in $m.s^{-1}$.
- Acceleration, $\vec{a}$, measured in $m.s^{-2}$.

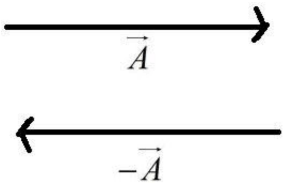
Vectors are shown by symbols with an arrow pointing to the right above it. For example, force is represented by the letter F with an arrow above it: $\vec{F}$. Velocity is represented like this: $\vec{v}$. This arrow DOES NOT indicate the direction of the vector. It is just the notation for representing a vector.

If two vectors have the same magnitude (size) and the same direction, then they are EQUAL. This is known as **equality of vectors**. The two vectors below are equal because they have the same magnitude and the same direction.



*[D] represents a definition. Definitions need to memorised word for word.

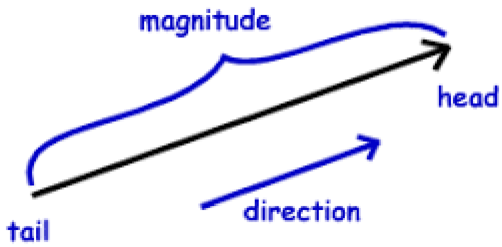
The two vectors below are not equal. Even though their magnitude is identical, their direction is opposite.



When working with vectors, we state a positive direction. Based on the above vectors, towards the right is the positive direction. Therefore, a negative vector will act to the left.

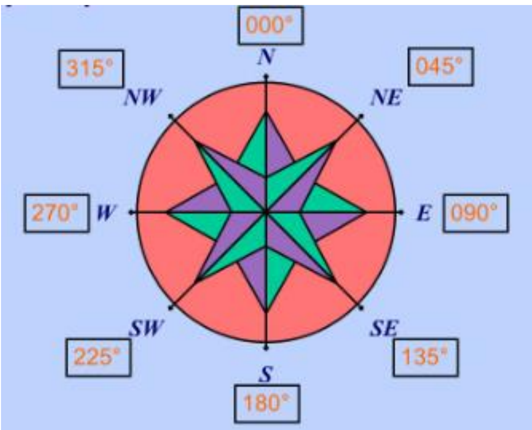
Representing vectors graphically

Vectors are drawn as arrows. An arrow has both a magnitude and direction. The magnitude is based on a scale and the length of the arrow. The direction is based on the direction of the arrowhead.

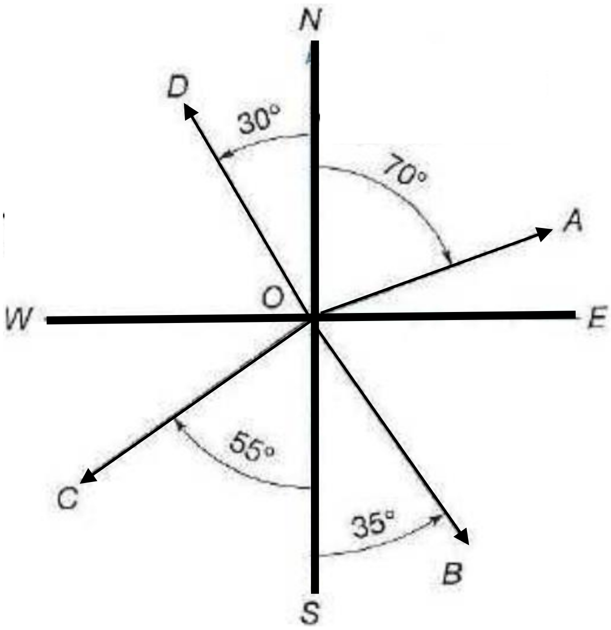


Defining the direction of a vector

When it comes to vectors, magnitude is only half of the answer. The direction makes up the other half. There are many acceptable methods of describing the direction of a vector e.g. left, right, up and down. These may also be expressed with arrows: ←, →, ↑ and ↓. Another common method of expressing directions is to use the points of a compass: north (N), east (E), south (S) and west (W). Exactly in the middle of N and E is northeast (NE). Exactly in the middle of S and E is southeast (SE). Exactly in the middle of S and W is southwest (SW). Exactly in the middle of N and W is northwest (NW).



Consider the vectors below (A, B, C and D) that are not pointing in one of the exact compass directions mentioned above. Note that this notation must begin with NORTH OR SOUTH, never east or west. The angle is therefore always North of East, North of West, South of East or South of West. In your mind, change the word **of** to **from**, and this will help you to identify the angle.



Vector A's direction is 20° North of East (because it is 20° towards the North **from** the East). This same angle may also be expressed as N 70° E.

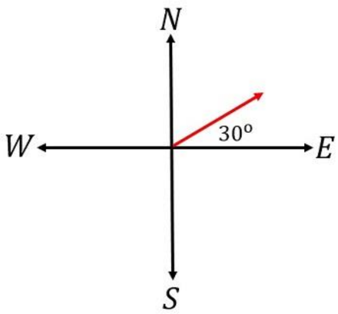
Vector B's direction is 55° South of East (because it is 55° towards the South **from** the East). This same angle may also be expressed as S 35° E.

Vector C's direction is 35° South of West (because it is 35° towards the South **from** the West). This same angle may also be expressed as S 55° W.

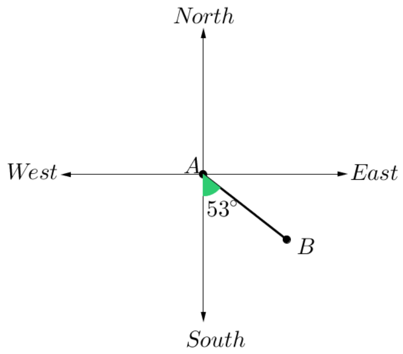
Vector D's direction is 60° North of West (because it is 60° towards the North **from** the West). This same angle may also be expressed as N 30° W.

Another method of expressing direction is to use the bearing method. A bearing is a direction relative to a fixed point – from North, moving clockwise. A bearing is always written as a three-digit number, for example 275° or 080° (for 80°).

The bearing of the vector below is 060°



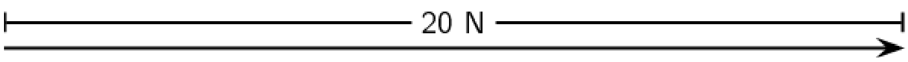
The bearing of the vector below is 127°



Using the head-to-tail method to represent vectors graphically

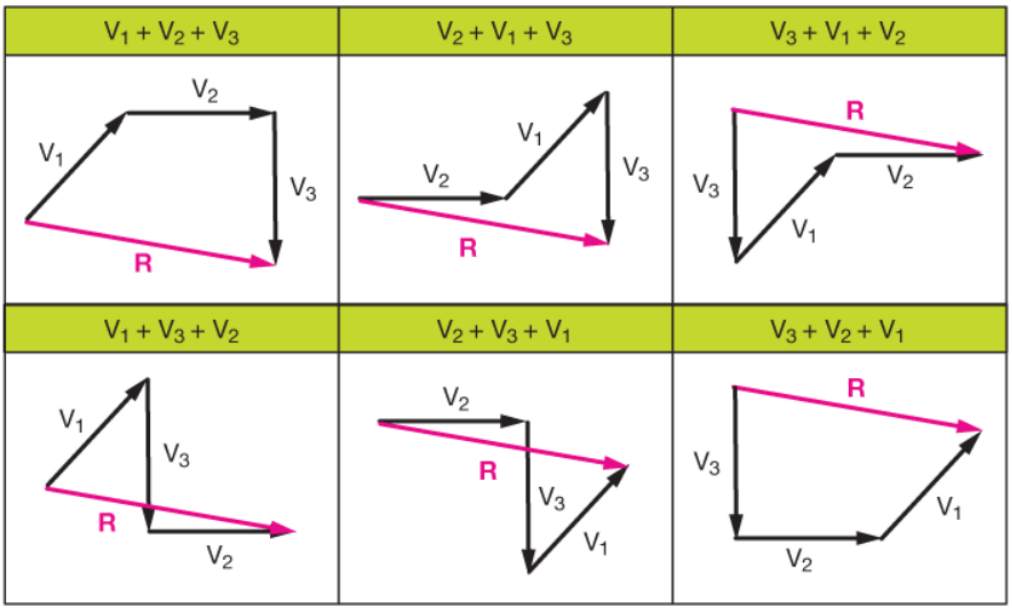
To draw a vector accurately we must represent its magnitude using a scale. A scale allows us to translate the length of the arrow into the vector's magnitude. If we draw a vector that represents 20 N to the east, we first need to choose a scale,
e.g. 1 cm = 2 N
∴ 10 cm = 20 N

Note that we label the magnitude of the vector (20 N), not the scale (10 cm).



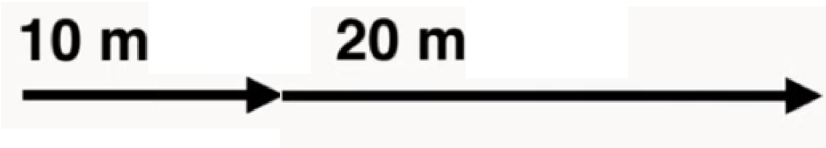
Graphical techniques involve drawing accurate scale diagrams to denote individual vectors and their resultants. We use the head-to-tail method. We always **start by choosing a scale**.

Note that the order in which we add the vectors does not influence the final answer. As can be seen in the diagram below, vectors 1, 2 and 3 are added in six different orders, yet the resultant vector, R, is the same in all cases:



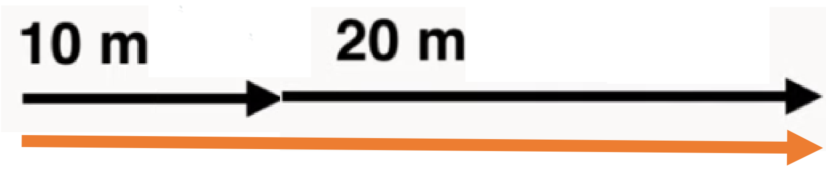
To use the head-to-tail method, draw the first vector using the scale and an arrowhead to denote direction.

Now take the next vector and draw it with the tail starting from the arrowhead of the first vector.



Once all the vectors have been added in this way, we can find the resultant vector **<u>from the tail of the first vector to the head of the last vector</u>**. The arrowhead is placed at the head of the last vector (the arrowhead of the resultant vector will always touch the arrowhead of the last vector). The magnitude of the resultant vector may now be identified based on the original scale.

A resultant vector [D] is single vector whose effect is the same as the individual vectors acting together.

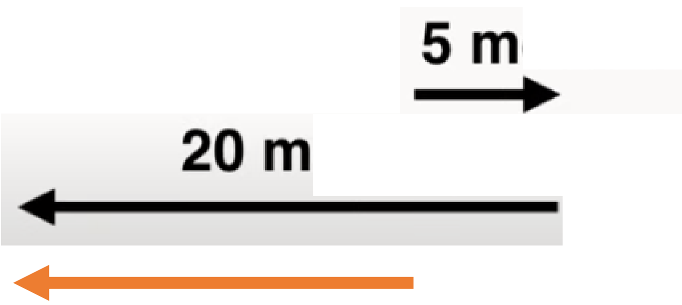


$$\vec{R} = 30 \text{ m}$$

Let's apply the head-to-tail method to an example where the vectors act in opposite directions:

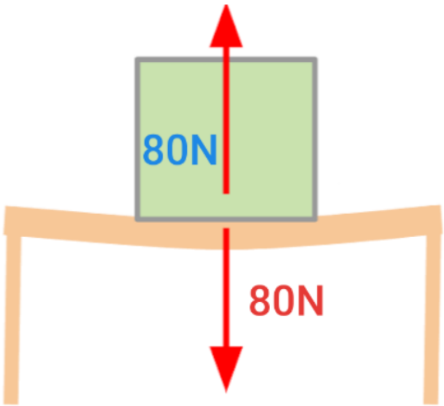
$$\vec{R} = 15 \text{ m}$$

OR

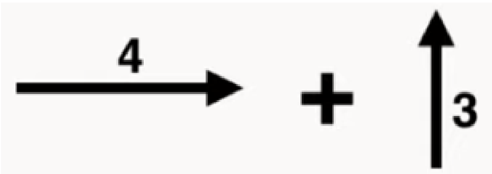


$$\vec{R} = 15 \text{ m}$$

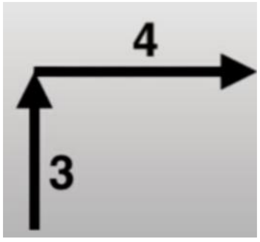
An **equilibrant** is a vector that has the same magnitude but is opposite in direction to the resultant vector. The purpose of the equilibrant is to bring a system into equilibrium by counteracting the resultant vector. For example, the resultant force of the box on the table is 80 N ↓. The equilibrant force is 80 N ↑. If you add the resultant vector and the equilibrant vectors together, the answer is always zero because the equilibrant cancels out the resultant force.



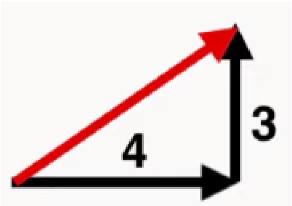
When using the head-to-tail method, we are not limited to vectors that are in one dimension. We may also add vectors in two dimensions. Below are two vectors that are at right angles to one another.



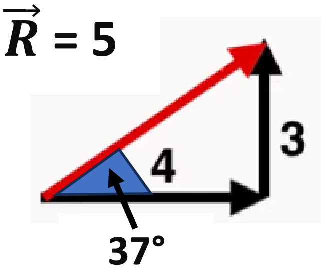
Choose a scale, and add the vectors based on their directions and the scale as follows. We need to use a protractor to ensure that the two vectors are at right angles to one another:



The resultant vector will look like this:

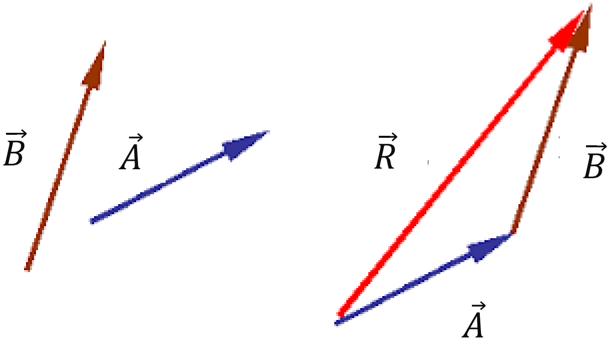


The magnitude of the vector may be identified using the scale, and the direction may be identified using a protractor. We always **measure the direction of the resultant vector from the tail of the resultant vector**. The easiest method for stating direction is the bearing method. So from the tail of the resultant vector, **state the direction based on the bearing method**.

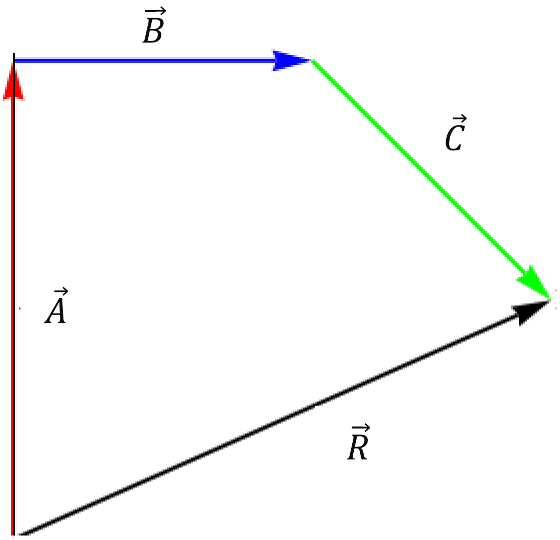


The resultant vector is 5 (units) at a bearing of 053°.

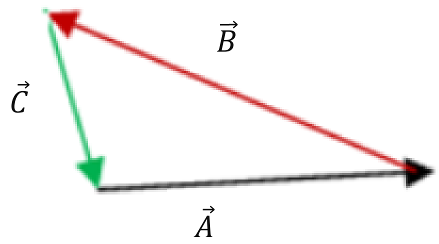
Two dimensional vectors are not necessarily at right angles to one another. Consider vectors $\vec{A}$ and $\vec{B}$ below. The resultant of $\vec{A}$ and $\vec{B}$ is represented by $\vec{R}$.



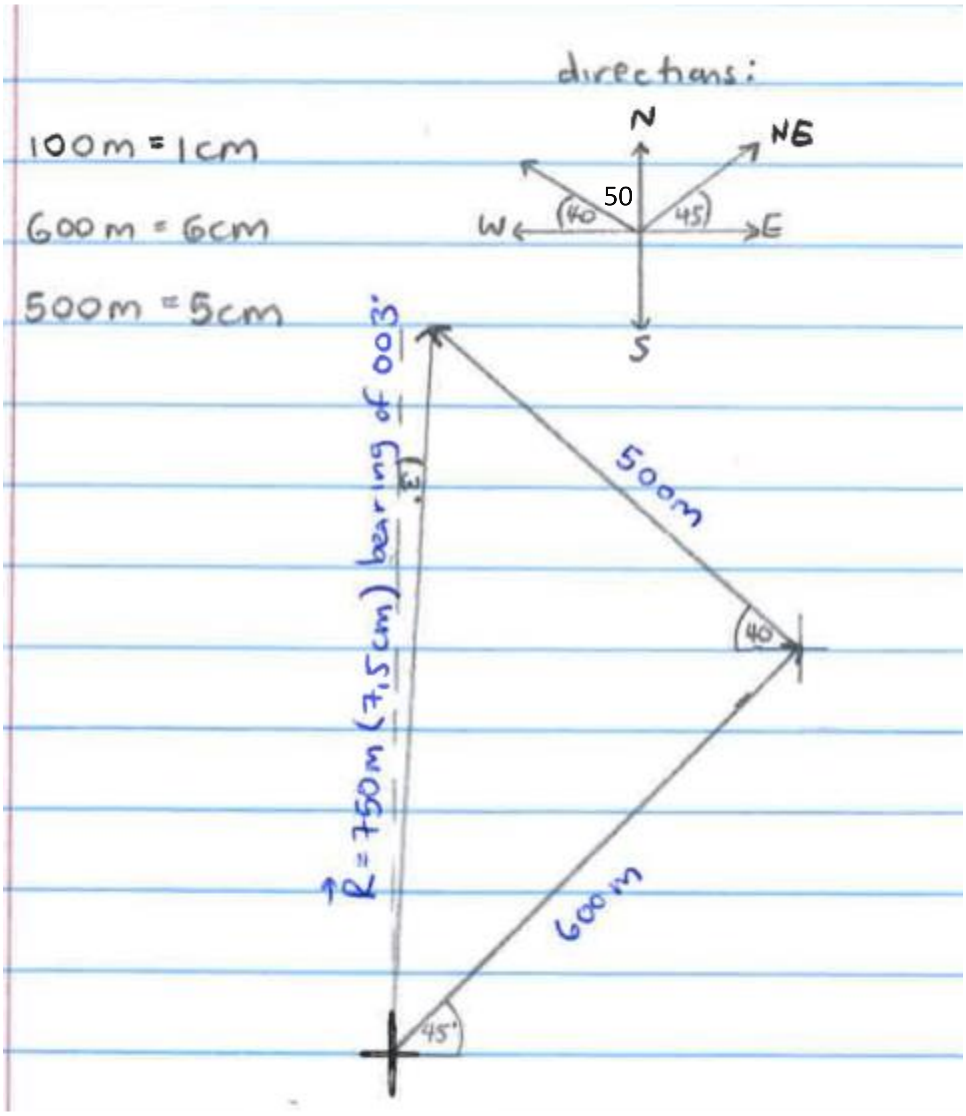
Consider vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ below. The resultant of $\vec{A}$, $\vec{B}$ and $\vec{C}$ is represented by $\vec{R}$.



Lastly, in a closed vector diagram, when all the vectors are added using the head-to-tail method, the tail first vector starts at the origin, and the head of the last vector ends at the origin. A polygon is created, meaning that the resultant vector has a magnitude of zero.



Sarah walks to school by walking 600 m northeast and 500 m N 50° W. Determine her resultant displacement. (Northeast is a 45° angle between N and E. N 40° W means a 40° angle towards the north measured from the west).



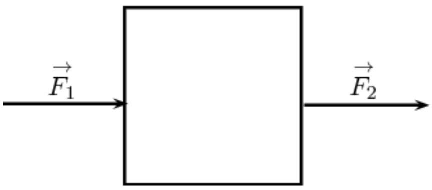
Note the following when drawing vector diagrams:

- Start off with a simple scale. Based on your scale, calculate the length of each vector in mm or cm.
- Make a small compass on your page and draw, not to scale, the directions of each vector, to help you to identify the direction of the angles.
- Now you're ready to start drawing your vector diagram.
- Make a small compass for your first vector anywhere on the page.
- Measure and draw in the correct angle and length of the first vector (see the 45° angle).
- Make small compass at the head of the first vector.
- From there, measure the correct angle (see the 40° angle) and draw the correct length of the second vector with the tail coming off the head of the first vector.
- Once all your vectors have been drawn using the head-to-tail method, you may find the resultant vector.
- Connect the resultant vector from the tail of the first vector to the head of the last vector.
- The direction of the resultant vector will be indicated by the arrowhead of the vector that will touch the arrowhead of the last vector.
- Measure the length of the resultant vector to identify the **magnitude** of the resultant vector using the scale.
- The length of the resultant vector is 7.5 cm, which, based on the scale, is 750 m.
- To identify the angle of the resultant vector, we need to measure it from **the tail of the resultant** vector.
- Make a small compass at the tail of the resultant vector.
- From the tail of the resultant vector, measure an angle that will allow you to identify the direction of the resultant vector using the bearing method. That is, from the N, clockwise.
- In the vector diagram below, this is a particularly tricky angle to identify since the resultant vector is close to being a 90° angle. However, it is 3° from the N in a clockwise direction so the bearing is 003°.

Algebraic methods of vector addition

Addition and subtraction of vectors in one dimension.

You and a friend are trying to move a heavy box. You stand behind it and push forwards with a force of $\vec{F}_1 = 5\ N$. Your friend stands in front and pulls the box in the same direction with a force of $\vec{F}_2 = 3\ N$. Calculate the resultant force.



Note that in this case, the two forces are acting in the same direction – to the right. When vectors are added algebraically, we need to **start by stating a positive direction**.

Let the positive direction be to the right.

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2$$
$$= 5 + 3$$
$$= 8\ N\ to\ the\ right$$

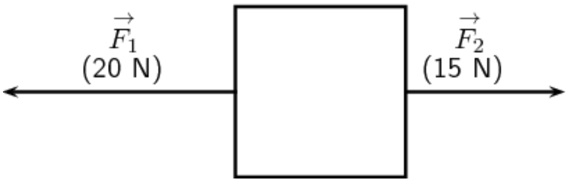
Note that a positive answer means that the direction is positive. A positive direction is to the right.

Alternatively, let the positive direction be to the left. This means that the forces are acting in the negative direction.

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2$$
$$= -5 - 3$$
$$= -8 = 8\ N\ to\ the\ right$$

Note that a negative answer means that the direction is negative. A negative direction is to the right. Regardless of which you choose as the positive direction, you will derive the same answer.

You are trying to move a heavy box. You pull the box to the left with a force of $\vec{F}_1 = 20\ N$. The box exerts a frictional force of $\vec{F}_2 = 15\ N$. Calculate the resultant force.



Let the positive direction be to the left.

$\vec{F}_R = \vec{F}_1 + \vec{F}_2$
$= 20 - 15$
$= 5\ N\ to\ the\ left$

Note that a positive answer means that the direction is positive. A positive direction is to the left.

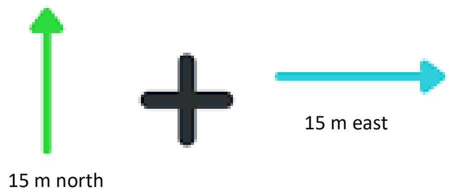
Alternatively, let the positive direction be to the right.

$\vec{F}_R = \vec{F}_1 + \vec{F}_2$
$= -20 + 15$
$= -5 = 5\ N\ to\ the\ left$

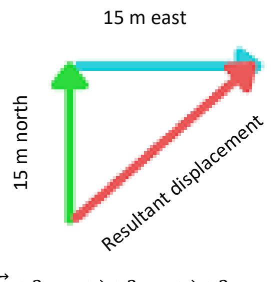
Note that a negative answer means that the direction is negative. A negative direction is to the left.

Algebraic technique for vectors at right angles (two dimensions)

Given two vectors that act at a right angel to one another, we may solve the length of the resultant (hypotenuse) side using the Pythagoras theorem. Shayna walks to school first by traveling 15 m north ($\vec{x}_1 = 15\ m\ N$), and then 15 m east ($\vec{x}_2 = 15\ m\ E$). What is her resultant displacement?



In this case we draw a sketch, but not to scale. The sketch is used to help us see the vector directions, and assess the direction of the resultant vector.



$(\overrightarrow{R_x})^2 = (\vec{x}_1)^2 + (\vec{x}_2)^2$

$\therefore (\vec{R}_x)^2 = 15^2 + 15^2$

$\therefore (\vec{R}_x)^2 = 225 + 225$

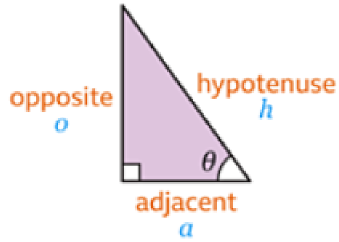
$\therefore (\vec{R}_x)^2 = 450$

$\therefore \vec{R}_x = \sqrt{450}$

$\therefore \vec{R}_x = 21.21\ m$

This is the magnitude of the resultant displacement.

For vectors, we also need to state direction. For this, we use trigonometry:



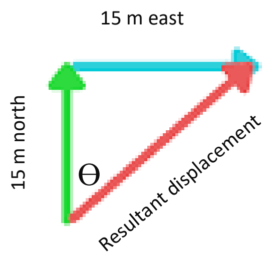
Where:

$$\sin\Theta = \frac{opposite}{hypotenuse}$$

$$\cos\Theta = \frac{adjacent}{hypotenuse}$$

$$\tan\Theta = \frac{opposite}{adjacent}$$

Recall that the angle is always measured from the tail of the resultant. Note the placement of angle ϴ:



$\tan\theta = \frac{opposite}{adjacent}$

$\therefore \tan\theta = \frac{15}{15}$

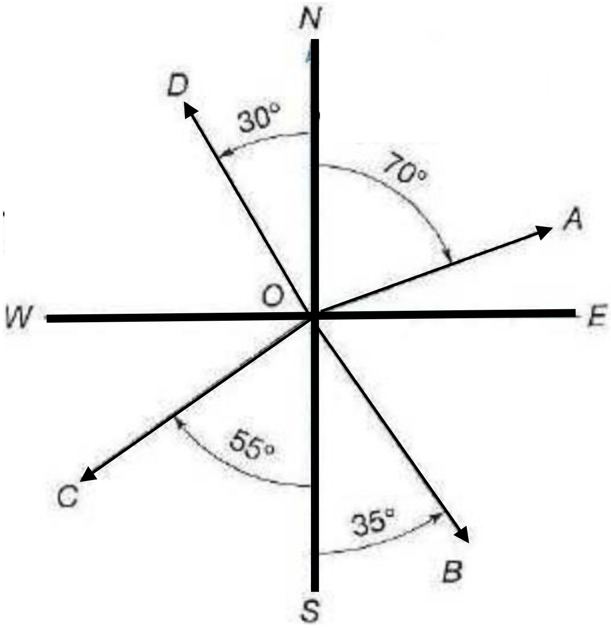
$\therefore \tan\theta = 1$

$\therefore \theta = 45°$

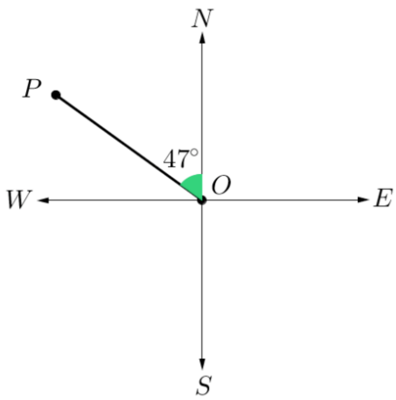
Therefore, Shayna's displacement is 21.21 m at a bearing of 045°.

Practice questions

1. Consider the diagram below.
   1.1 What is the direction of vector C?
   1.2 What is the direction of vector D?



2. What is the bearing of the vector below?



3. Calculate the resultant vector **graphically**: A squash ball is dropped to the floor with an initial velocity of 2.5 $m.s^{-1}$. It rebounds (comes back up) with a velocity of 1.2 $m.s^{-1}$. Determine the resultant velocity of the ball.

4. Calculate the resultant vector **graphically**: A man applies a force of 5 N on a crate. Another man applies a force of 7 N on the crate in the same direction. The crate exerts a frictional force of 2 N. Determine the resultant force on the crate.

5. Calculate the resultant vector **graphically**: A frog is trying to cross a river. It swims at 3 $m.s^{-1}$ in a northernly direction towards the opposite bank. The water is flowing in a westerly direction at 5 $m.s^{-1}$. Find the frog's resultant velocity.

6. Calculate the resultant vector **graphically**: A dove flies from her nest in search of food. She flies at a velocity of 2 $m.s^{-1}$ at a bearing of 135° and then at a velocity of 1.5 $m.s^{-1}$ at a bearing of 230°. Calculate her resultant velocity.

7. Calculate the resultant vector **algebraically**: A squash ball is dropped to the floor with an initial velocity of 2.5 $m.s^{-1}$. It rebounds (comes back up) with a velocity of 1.2 $m.s^{-1}$. Determine the resultant velocity of the ball.

8. Calculate the resultant vector **algebraically**: A man applies a force of 5 N on a crate. Another man applies a force of 7 N on the crate in the same direction. The crate exerts a frictional force of 2 N. Determine the resultant force on the crate.

9. Calculate the resultant vector **algebraically**: A frog is trying to cross a river. It swims at 3 $m.s^{-1}$ in a northernly direction towards the opposite bank. The water is flowing in a westerly direction at 5 $m.s^{-1}$. Find the frog's resultant velocity.

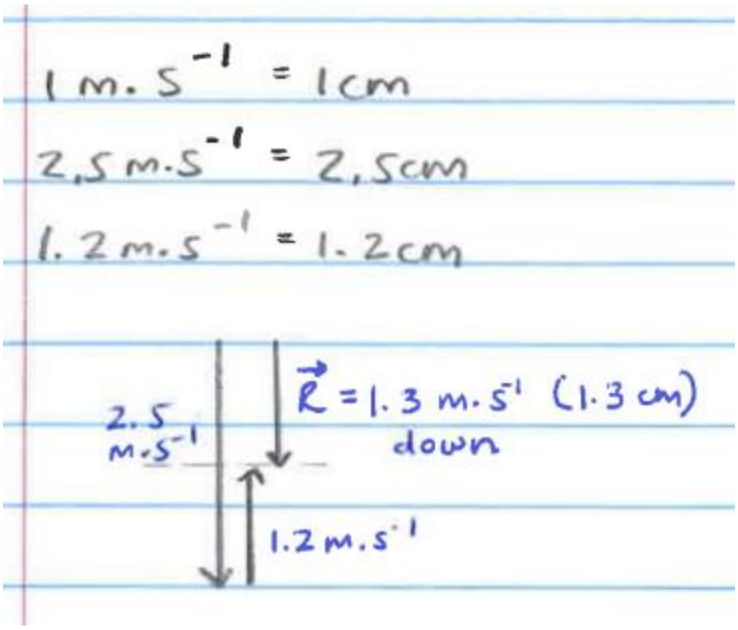
10. Calculate the resultant vector **algebraically**: A plane is traveling with a velocity of 100 $km.h^{-1}$ south. It encounters a side wind of 25 $km.h^{-1}$ west. Determine the resultant velocity of the plane.

Practice question answers

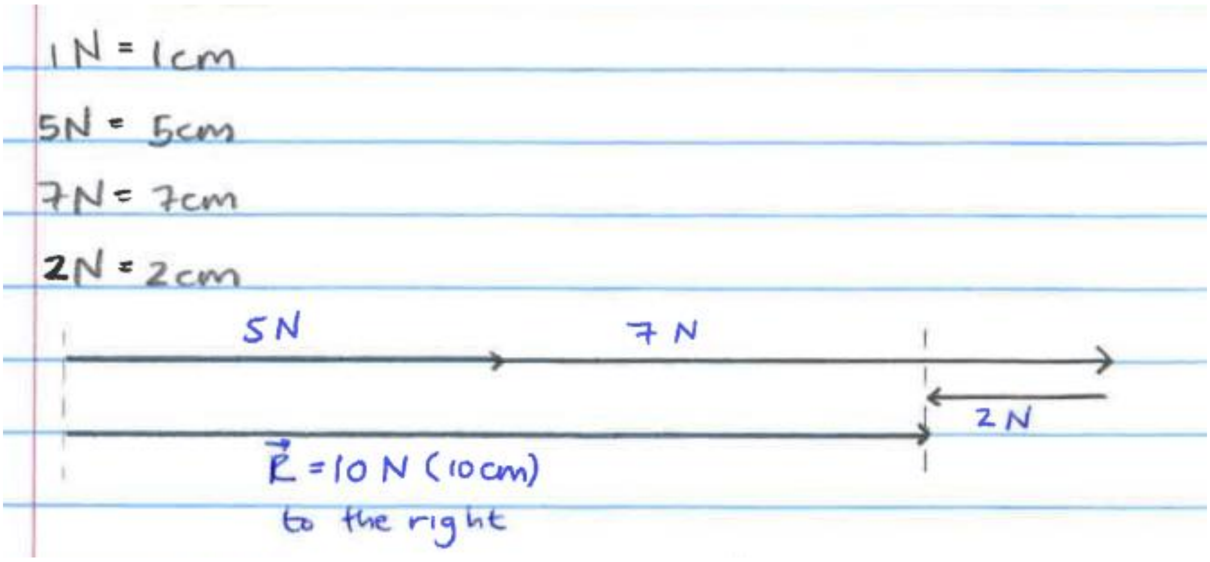
1.
   1.1 35° South of West, or S 55° W.
   1.2 60° North of West, or N 30° W.

2. Bearing of 313°.

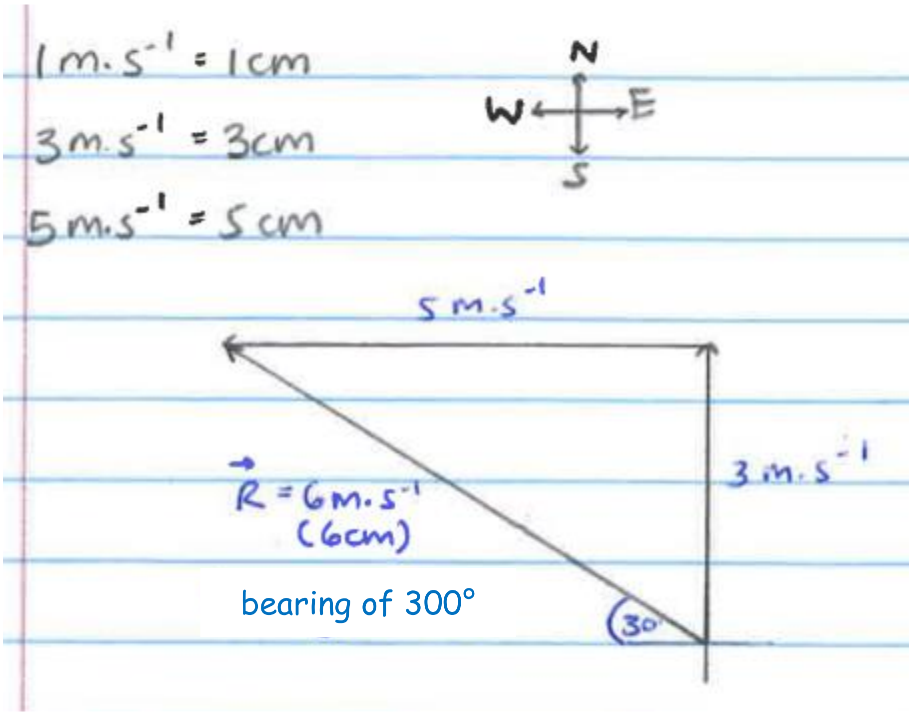
3.



4.

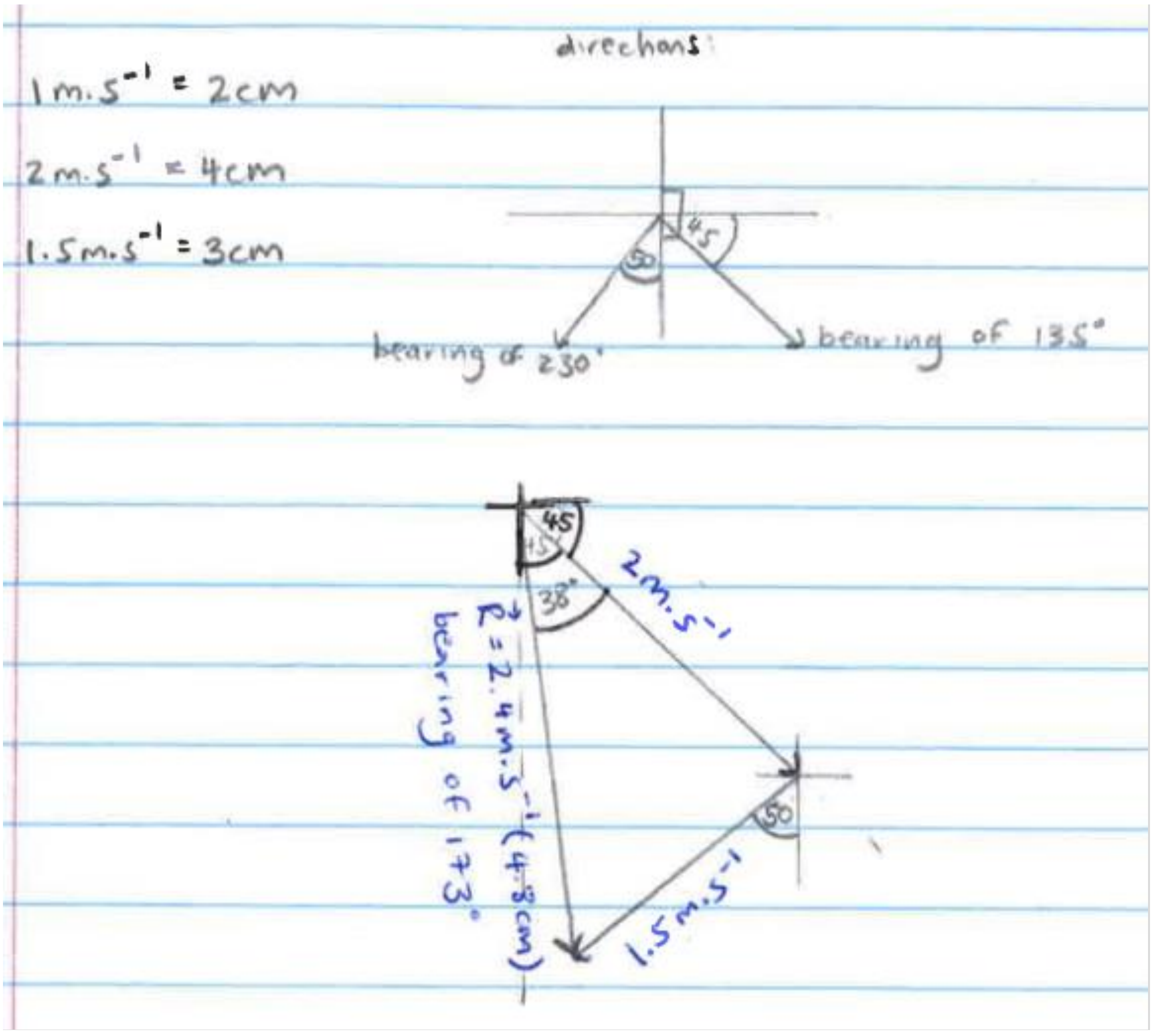


5.



1 m.s⁻¹ = 1cm
3 m.s⁻¹ = 3cm
5 m.s⁻¹ = 5 cm
N
W
E
S
5 m.s⁻¹
3 m.s⁻¹
R = 6m.s⁻¹ (6cm)
bearing of 300°
30°

6.



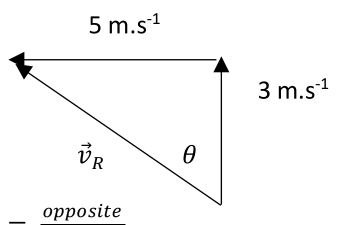
directions:
1 m.s⁻¹ = 2cm
2 m.s⁻¹ = 4cm
1.5 m.s⁻¹ = 3cm
45
50
bearing of 230°
bearing of 135°
45
45
38°
2 m.s⁻¹
R = 2.4 m.s⁻¹ (4.8cm)
bearing of 173°
50
1.5 m.s⁻¹

7. Let down be the positive direction.
$\vec{v}_R = \vec{v}_1 + \vec{v}_2$
$= 2.5 + (-1.2)$
$= 1.3\ m.s^{-1}\ down$

8. Let the direction of the force applied by the men be the positive direction.
$\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$
$= 7 + 5 + (-2)$
$= 10\ N\ in\ the\ direction\ of\ the\ force\ applied\ by\ the\ men$

9. $(\vec{v}_R)^2 = (\vec{v}_1)^2 + (\vec{v}_2)^2$
$\therefore (\vec{v}_R)^2 = 5^2 + 3^2$
$\therefore (\vec{v}_R)^2 = 25 + 9$
$\therefore (\vec{v}_R)^2 = 34$
$\therefore \vec{v}_R = \sqrt{34}$
$\therefore \vec{v}_R = 5.83\ m.s^{-1}$

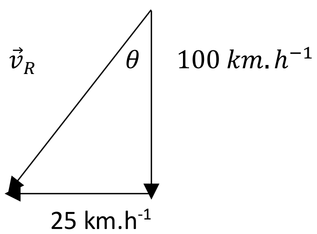
To identify the direction, we need a sketch, which is a rough drawing not drawn to scale. The angle is placed at the tail of the resultant vector:



$\tan\theta = \frac{opposite}{adjacent}$
$\therefore \tan\theta = \frac{5}{3}$
$\therefore \tan\theta = 1.67$
$\therefore \theta = 59°$
The bearing is therefore $360° - 59° = 301°$
The frog's resultant velocity is 5.83 $m.s^{-1}$ at a bearing of 301°.

10. $(\vec{v}_R)^2 = (\vec{v}_1)^2 + (\vec{v}_2)^2$
$\therefore (\vec{v}_R)^2 = 100^2 + 25^2$
$\therefore (\vec{v}_R)^2 = 10000 + 625$
$\therefore (\vec{v}_R)^2 = 10625$
$\therefore \vec{v}_R = \sqrt{10625}$
$\therefore \vec{v}_R = 103.08\ km.h^{-1}$

To identify the direction, we need a sketch. The angle is placed at the tail of the resultant vector:



$\tan\theta = \frac{opposite}{adjacent}$
$\therefore \tan\theta = \frac{25}{100}$
$\therefore \tan\theta = 0.25$
$\therefore \theta = 14.04°$
The bearing is therefore $180° + 14.04° = 194.04°$
The plane's resultant velocity is $103.08\ km.h^{-1}$ at a bearing of $194.04°$.